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$MA(\sigma$-CENTERED): COHEN REALS, STRONG MEASURE ZERO SETS AND STRONGLY MEAGER SETS

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ABSTRACT

We prove that $MA(\sigma$-centered) + the Dual Borel Conjecture is consistent; and that $MA(\sigma$-centered) + the non-additivity of the ideal of the strong measure zero sets also is consistent.

§0. Introduction

In this work we will study models for set theory where $MA(\sigma$-centered) holds. This extra assumption for the set theory universe has been studied by many people, and it is known, for example, that it is also consistent with the non-existence of random reals over L. The results in Sections §1, 2 are a strongest form of the above-mentioned fact about random reals, and in order to state them, we need some definitions.

0.1. Definition (Borel). We say that $X \subseteq \mathbf{R}$ has strong measure zero if for every $\langle \varepsilon_i : i < \omega \rangle \subseteq \mathbf{R}^+$ there is $\langle x_i : i < \omega \rangle$ such that

$$X \subseteq \bigcup_{i<\omega} (x_i - \varepsilon_i, x_i + \varepsilon_i).$$

0.2. Theorem (Carlson). *If the union of κ-many measure zero sets is a measure zero set then the union of less than κ-many strong measure zero sets is a strong measure zero set.*

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F. Galvin asked if in the Carlson theorem it is possible to replace, from the hypothesis, "measure zero' by meager.

In the first section we will show that this is impossible by showing

0.3. THEOREM. *If ZF is consistent then there is a model for ZFC satisfying*

(i) $MA(\sigma\text{-centered}) + 2^{\aleph_0} \geqq \aleph_2$.

(ii) *There is* $\langle X_i : i < \omega_1 \rangle$, *each* X_i *is a strong measure zero set but* $\bigcup X_i$ *is not a strong measure zero set.*

Some time ago, Galvin, Mycielski and Solovay gave a very elegant description of the strong measure zero sets, namely

0.4. THEOREM (Galvin–Mycielski–Solovay). $X \subseteq \mathbf{R}$ *has strong measure zero iff for every meager set* M *there is* $x \in \mathbf{R}$ *such that*

$$(x + X) \cap M = \varnothing. \qquad \blacksquare$$

Using this characterization, we naturally consider the following objects.

0.5. DEFINITION. $X \subseteq \mathbf{R}$ is strongly meager if for every Lebesgue measure zero set M there is $x \in \mathbf{R}$ such that $(X + x) \cap M = \varnothing$. ■

These sets are very unfriendly and we know very little about them. For instance, we don't know if the collection of strongly meager sets is an ideal. Anyway, we can state the Dual Borel Conjecture.

0.6. DEFINITION. The Dual Borel Conjecture says that every strongly meager set is countable.

Carlson proved the consistency of the Dual Borel Conjecture.

0.7. THEOREM (Carslon). *If ZF is consistent then there is a model for ZFC + the Dual Borel Conjecture.* ■

Carlson's Theorem is very strong. His proofs show that every model might be extended to a model for the Dual Borel Conjecture.

In Section §2 we will show that $MA(\sigma$-centered) does not produce uncountable strongly meager sets by showing

0.8. THEOREM. *If ZF is consistent then there is a model for ZFC + the Dual Borel Conjecture +* $MA(\sigma\text{-centered})$.

S. Todorcevic points out a discrepancy between a result of Roitman [9] on one hand and a consequence of a result of Shelah [10] §2 on the other hand. We will use Carlson's Theorem 0.6 in order to see that the result in Roitman [9] is false.

0.9. THEOREM. *If M is a model of ZFC and $2^{\aleph_0} > \aleph_1$ and r is a Cohen real over M then $M[r] \models$ "$MA(\sigma$-linked) fails".*

PROOF. Let M be a model such that if r is Cohen real over M then $M[r] \models MA(\sigma$-linked). Let r_{ω_2} be ω_2-Cohen reals over M. Then by Carlson [2],

$$M[r_{\omega_2}] \models \text{"Dual Borel Conjecture"}.$$

But for every real $a \in M[r_{\omega_2}]$ there exists a Cohen real $r \in M[r_{\omega_2}]$ such that $a \in M[r]$, and because $M[r] \models$ "$MA(\sigma$-linked)", we have in $M[r_{\omega_2}]$ that the following holds: let $X \in [\mathbf{R}]^{\omega_1} \cap M$. If A is a Borel null set of reals then there exists a real number b such that

$$(b + X) \cap A = \emptyset$$

(using a random real over $L[X]$). This fact implies that

$$M[r_{\omega_2}] \models \text{"Dual Borel Conjecture fails"}. \qquad \blacksquare$$

However, some part of MA may hold after adding a Cohen real, namely

0.10. THEOREM. *If $M \models$ "$MA(\sigma$-centered)" and r is Cohen real over M, then $M[r] \models$* "**MA**(*σ-centered*)".

This theorem was proved independently by many people, including the authors, and we include in this section a sketch of Baumgartner's proof. This proof uses Bell's theorem, ([0]), which says that $p = c$ is equivalent to $MA(\sigma$-centered).

Sketch that $p = c$ is preserved by adding one Cohen real. Consider a Cohen real as a function from ω into 2. Let $\{\dot{a}_\alpha : \alpha < \kappa\}$, where $\kappa < c$, be a set of terms for a family which is forced to have f.i.p. For $k \in \omega$, define a k-packet to be a sequence $\{(p_\sigma, n_\sigma) : \sigma \in {}^k 2\}$ where p_σ is a Cohen condition, and $\sigma \leqq p_\sigma$. Let A_α be the set of all k-packets $\{(p_\sigma, n_\sigma) : \sigma \in {}^k_2\}$ such that $p_\sigma \Vdash n_\sigma \in \dot{a}_\alpha$, and k varies over all positive integers. let $B_n = \{k\text{-packets} : k \geqq n\}$. Then $\{B_n : n < \omega\} \cup \{A_\alpha : \alpha < \kappa\}$ has f.i.p., therefore it has a lower bound A. If $\dot{a}$ is the term defined by $p \Vdash n \in \dot{a}$ iff for some $q \geqq p$, $(q, n) \in \bigcup A$, then (proof should be provided by the reader) $\dot{a} \subset^* \dot{a}_\alpha$ for all α. $\blacksquare$

The authors are grateful to the referee for his remarks improving the

presentation of this work. The Baumgartner proof was also supplied by the referee.

All notations used in this article are standard, and a good reference is Kunen [5], where it is possible to find the definition of the forcing notions used in this work (i.e. Cohen real forcing, etc.).

§1. Strong measure zero sets

We recall

1.1. Definition. $X \subseteq \mathbf{R}$ has strong measure zero if for every $\langle \varepsilon_i : i < \omega \rangle$ $(\varepsilon_i > 0)$ there exists $\langle x_i : i < \omega \rangle \subseteq \mathbf{R}$ such that $X \subseteq \bigcup_{i<\omega}(x_i - \varepsilon_i, x_i + \varepsilon_i)$.

1.2. Fact. Clearly $\{X : X$ has strong measure zero$\}$ is a σ-ideal.

1.3. Definition. (a) Let I be a σ-ideal of subsets of reals. Then Add(I) holds if and only if for every $\langle A_\alpha : \alpha < \kappa \rangle$ such that $\kappa < 2^{\aleph_0}$ and each $A_\alpha \in I$, $\alpha < \kappa$, we have that $\bigcup_{\alpha<\kappa} A_\alpha \in I$.

(b) Let

$$\mathscr{B} = \{A \subseteq \mathbf{R} : A \text{ is a meager set}\},$$

$$\mathscr{L} = \{A \subseteq \mathbf{R} : A \text{ has Lebesgue measure zero}\},$$

$$SMZ = \{A \subseteq \mathbf{R} : A \text{ has strong measure zero}\}.$$

1.4. Theorem (Bartoszynski). Add($\mathscr{L}$)$\Rightarrow$Add($\mathscr{B}$). ■

1.5. Theorem (Carlson). $MA(\mathscr{L}) \Rightarrow$ Add(SMZ).

We will give the proof of Carlson's theorem. We will show something stronger. Instead of working in $\mathbf{R}$ we will work in 2^ω. The following is easy to show

1.6. Fact. $X \subseteq 2^\omega$ has strong measure zero if and only if for every $f \in \omega^\omega$ (w.l.o.g. f is increasing) there exists $g \in (2^{<\omega})^\omega$ such that

(i) for each $n \in \omega$, $g(n) \in 2^{f(n)}$,

(ii) for every $h \in X$ there exists infinitely many $n \in \omega$ such that $h \restriction f(n) = g(n)$.

(In other words, if $[g(n)] = \{h \in 2^\omega : h \restriction f(n) = g(n)\}$ then condition (ii) says $X \subseteq \bigcap_{n<\omega} \bigcup_{m \geq n} [g(m)]$.)

1.7. Fact (Bartoszynski). Add($\mathscr{L}$) iff for every $F \in [\omega^\omega]^{<c}$ there exists $h \in (\omega^{<\omega})^\omega$ such that

(i) $(\forall n \in \omega)(|h(n)| = n)$,

(ii) $(\forall f \in F)(\exists n \in \omega)(\forall m \geqq n)(f(n) \in h(n))$.

PROOF. See [1]. ■

Now we are ready to show Carlson's Theorem: Let $\langle X_i : i < \kappa \rangle$, $\kappa < c$ be strong measure zero sets. Let $f \in \omega^\omega$ be increasing. Let $g(n) = n \cdot f(n)$. Let $h_i \in (2^{<\omega})^\omega$ be such that for each $n \in \omega$, $h_i(n) \in 2^{g(n)}$ and $X_i \subseteq \bigcap_{n \in \omega} \bigcup_{m \geqq n} [h_i(n)]$. Using 3.7 we let $h \in ((2^{<\omega})^{<\omega})^\omega$ such that for each $n \in \omega$, $|h(n)| = n$ and for each $i < \kappa$ there is $n \in \omega$ such that for every $m \geqq n$

$$h_i(m) \in h(m).$$

Now it is not hard to see that

(i) $\bigcup_{i<\omega} X_i \subseteq \bigcap_{n \in \omega} \bigcup_{s \in h(m)} [s]$,

(ii) $\mu(\bigcup_{s \in h(n)} [s]) = n \cdot 2^{-g(n)} \leqq 2^{-f(n)}$.

This completes the proof of 1.5. Note that we are using only Add($\mathscr{L}$).

Many people, among them the first author, have conjectured that Add($\mathscr{B}$) already implies Add(SMZ). We will see that this is false, as can be deduced from the following two theorems.

1.8. THEOREM (Folklore). MA(*σ-centered*)$\Rightarrow$Add($\mathscr{B}$). ■

1.9. THEOREM. Cons(ZF)$\Rightarrow$Cons($ZFC + MA(\sigma$-centered$) + \neg$Add(SMZ)).

The proof of this theorem will take the rest of this section. We need some definitions and technical lemmas.

1.10. DEFINITION. (a) Let $f \in \omega^\omega$, and $\bar{\eta} = \langle \eta_l : l \in a \rangle$, $a \in [\omega]^\omega$. Then we say that $\bar{\eta}$ obeys f if and only if for every $l \in a$, $\eta_l \in 2^{f(l)}$ and for every $q \in a$ there exist infinitely many $p \in \omega$ such that

$$\eta_q \subseteq \eta_p$$

(we will assume $f(l+1) > 2^{2^{f(l)}}$).

(b) Let $\bar{\nu} = \langle \nu_l : l \in a \rangle$, $\bar{\eta} = \langle \eta_m : m \in b \rangle$ be given, then we define

$$\bar{\eta} <^* \bar{\nu}$$

if and only if for every $l \in \omega$ there exists $k \in \omega$ such that for every $p \in a - k$ there is $m \in b - l$ satisfying

$$\eta_m \subseteq \nu_p.$$

(c) Let $\bar{\eta} = \langle \eta_l : l \in a \rangle$ be given, then we define

$$A(\bar{\eta}) = \bigcap_{j\in\omega} \bigcup_{j<i\in a} [\eta_i].$$

1.11. Fact. Let $\bar{\nu}$ and $\bar{\eta}$ be given, then

$$\bar{\eta} <^* \bar{\nu} \Rightarrow A(\bar{\nu}) \subseteq A(\bar{\eta}).$$

Proof. If $x \in A(\bar{\nu})$ then there exists infinitely many $i \in a$ such that $x \upharpoonright f(i) = \nu_i$, for some function f. Fix $l \in \omega$ then there exists $k \in \omega$ such that for every $p \in a - k$ there is $m \in b - l$ satisfying

$$\eta_m \subseteq \nu_p.$$

So pick $i \in a - k$ s.t. $x \upharpoonright f(i) = \nu_i$. So pick $m \in b - l$ s.t. $\eta_m \subseteq \nu_p$, therefore $x \upharpoonright |\eta_m| = \eta_m$, and as l was arbitrary we have that $x \in A(\bar{\eta})$.

1.12. Definition. We say that $\langle f_i : i < \kappa \rangle$ is a dominating family if
(i) for every $i < \kappa$, $f_i \in \omega^\omega$;
(ii) for every $f \in \omega^\omega$ there exists $i < \kappa$ such that $f \leqq^* f_i$ (that is $(\exists m \in \omega\ \forall n \geqq m)(f(n) \leqq f_i(n))$).

1.13. Fact. (a) $X \subseteq 2^\omega$ has strong measure zero if and only if for every $f \in \omega^\omega$ there exists $\bar{\eta}$ obeying f such that $X \subseteq A(\bar{\eta})$.
(b) Let $\langle f_i : i < \kappa \rangle$ be a dominating family. Then X has strong measure zero iff for every $i < \kappa$ there exists $\bar{\eta}$ obeying f_i such that $X \subseteq A(\bar{\eta})$. ■

Sketch of the Forcing Construction. We will give a finite support iterated forcing, of length ω_2, $P_{\omega_2} = \underrightarrow{\mathrm{Lim}}\langle P_i; \underset{\sim}{Q}_i : i < \omega_2 \rangle$. Let

$$A = \{\alpha < \omega_2 : \mathrm{cof}(\alpha) = \omega_1\} \cup \{0\}.$$

Then we will have the following situation:

$$\alpha \in A \text{ then } \Vdash_{P_\alpha} \text{“}\underset{\sim}{Q}_\alpha\text{-adds a dominating real } \underset{\sim}{f}_\alpha\text{”}.$$

Therefore in the final extension $\langle f_\alpha : \alpha \in A \rangle$ will be a dominating family.

On the set $B = \{\alpha + 1 : \alpha \in A\}$ we will have the following situation:

$$\beta \in B \text{ then } \Vdash_{P_\beta} \text{“}\langle \underset{\sim}{x}_i^\beta : i < \omega_1 \rangle \text{ is a set of } \omega_1 \text{ reals”}.$$

In the final extension we will have
(a) For each $i < \omega_1$, $\{x_i^\beta : \beta \in \omega_2\} = X_i$ is of strong measure zero.
(b) $\bigcup_{i<\omega_1} X_i$ is not strong measure zero.

In order to get (b) we will do the construction in such a way that for every $\bar{\eta}$

which obeys f_0, there are some β and i satisfying $x_i^\beta \notin A(\bar{\eta})$. For this we need to give more definitions and some theorems about finite support iterated forcing.

Lastly we will get $MA(\sigma$-centered) because we have that $\omega_2 \setminus A \cup B \in [\omega_2]^{\omega_2}$ and we have enough room in order to pass for every σ-centered partial order coded in some intermediate stage.

Now we begin with some definitions and facts. The next definition plays a central role in the forcing construction.

1.14. DEFINITION. Let $\langle\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ be such that for every $i < \omega_1$, $\delta < \beta$, $\bar{\eta}^{\delta,i}$ obeys $f_\delta \in \omega^\omega$, and for $\delta_1 < \delta_2$ we have $f_{\delta_1} <^* f_{\delta_2}$. We say $\langle\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ is big if and only if for every $\bar{\eta}$ obeying f_0, there is $j < \omega_1$ such that for every $i > j$ and $\delta_1 \leqq \delta_2 < \beta$, there is no $q \in \omega$ satisfying

$$(\forall p)(q < p < \omega \wedge \eta_q^{\delta_1,i} \subset \eta_p^{\delta_2,i} \rightarrow (\exists s \in \omega)(\eta_q^{\delta_1,i} \subset \eta_s \subseteq \eta_p^{\delta_2,i})).$$

1.15. FACT. Let $\langle\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ be big. Let $\bar{\eta}$ be given obeying f_0. Then there exists $j \in \omega$ such that for every $i > j$, $\delta < \beta$ we have

$$A(\bar{\eta}^{\delta,i}) - A(\eta) \neq \varnothing.$$

PROOF. Because $\langle\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ is big we can find $j \in \omega_1$ satisfying 1.14 for $\bar{\eta}$. It is enough to show

$$A(\bar{\eta}^{\delta,i}) - A(\bar{\eta}) \neq \varnothing, \qquad \text{when } i > j.$$

Fix $q_0 \in \operatorname{dom}(\bar{\eta}^{\delta,i})$. Then there is $p_1 > q_0$ such that for no $s \in \omega$ we have $\eta_{q_0}^{\delta,i} \subset \eta^s \subseteq \eta_{p_1}^{\delta,i}$. Set $q_1 = p_1$, then there is $p_2 > q_1$ such that for no $s \in \omega$ we have $\eta_{q_1}^{\delta,i} \subset \eta^s \subseteq \eta_{p_2}^{\delta,i}$.

Continuing in this form we can get a sequence

$$\eta_{q_0}^{\delta,i} \subset \eta_{q_1}^{\delta,i} \subset \eta_{q_2}^{\delta,i} \subset \cdots.$$

Set $x = \bigcup_{j<\omega} \eta_{q_j}^{\delta,i}$. Then $x \in A(\bar{\eta}^{\delta,i}) - A(\bar{\eta})$. ∎

REMARK. (a) We will use this fact in order to add some x belonging to $A(\bar{\eta}^\alpha) - A(\bar{\eta})$. This will say that $A(\bar{\eta})$ is not a covering for the $\bigcup_{i<\omega_1} X_i$.

(b) In the construction we will assume that the family $\langle\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ has the following extra assumption: for $\delta_1 < \delta_2 < \beta$, for each $i < \omega_1$

$$\bar{\eta}^{\delta_1,i} <^* \bar{\eta}^{\delta_2,i}.$$

Therefore if $x \in A(\bar{\eta}^{\delta_2,i})$ then $x \in A(\bar{\eta}^{\delta_1,i})$.

The next goal is to show that the property of being big is preserved under the following conditions:

(a) Extending the universe by a σ-centered forcing notion (we will show something stronger).

(b) By finite support iteration.

These two properties are the basic technology for our construction.

1.16 LEMMA. *Let V be a model of ZFC and $\langle\langle\bar{\eta}^{\delta,i} : i < \omega_1\rangle : \delta < \beta\rangle$ is big in V. Let $P \in V$ be a forcing notion satisfying*:

$$(*)\qquad (\forall X \subseteq P)(|X| \geqq \aleph_1 \Rightarrow (\exists Y \subseteq X)(|Y| = \aleph_1 \wedge Y \text{ is directed})).$$

Then $\Vdash_P$ "$\langle\langle\bar{\eta}^{\delta,i} : \delta < \beta\rangle$ *is big*".

PROOF. Suppose that the conclusion of the lemma is false. Then there exists $\langle p_\alpha : \alpha \in A\rangle \subseteq P$ and $\underset{\sim}{\bar{\eta}} \in V^P$ and $p \in P$ satisfying

(i) $p \Vdash$ "$\underset{\sim}{\bar{\eta}}$ is witnessing $\langle\langle\eta^{\delta,i} : \delta < \beta\rangle$ is not big" (we are assuming $p \Vdash \underset{\sim}{\bar{\eta}}$ obeys f_0).

(ii) $|A| = \aleph_1$ and for every $\alpha \in A$, $p \leqq p_\alpha$.

(iii) For every $\alpha \in A$, there exists $k_\alpha \in \omega$ such that $\alpha < i_\alpha$ and

$$p_\alpha \Vdash \text{“}(\forall l)(k_\alpha < l < \omega \wedge \eta_{k_\alpha}^{\delta_\alpha^1, i_\alpha} \subset \eta_l^{\delta_\alpha^2, i_\alpha} \Rightarrow (\exists s \in \omega)(\eta_{k_\alpha}^{\delta_\alpha^1, i_\alpha} \subset \eta_s \subseteq \eta^{\delta_\alpha^2, i_\alpha}))\text{”}.$$

(iv) $(\forall \alpha_0, \ldots, \alpha_n \in \omega_1)(\exists q \in P)(\forall i \in [0, n])(p_{\alpha_i} \leqq q)$.

(v) Without loss of generality $k_\alpha = k$ and $\eta_k^{\delta_\alpha^1, i_\alpha} = \rho$.

Now we ask what are the possibilities for

$$\langle \underset{\sim}{\eta}_s : s \leqq l\rangle$$

for fixed l (w.l.o.g. $\mathrm{dom}(\underset{\sim}{\bar{\eta}}) = \omega$).

We have a finite number of such possibilities and each p_α, $\alpha \in A$, may contradict some part of them.

Let T_l be the set of $\langle \eta_s : s \leqq l\rangle$ such that for every $\alpha \in A$ there exists $q_\alpha \geqq p_\alpha$

$$q_\alpha \Vdash \langle \underset{\sim}{\eta}_s : s \leqq l\rangle = \langle \eta_s : s \leqq l\rangle.$$

1.17. FACT. (a) For every l, $T_l \neq \varnothing$. (Remember that A is directed and the family of possible $\langle \eta_s : s < l\rangle$ are finite.)

(b) If $\langle \eta_s : s \leqq l + 1\rangle \in T_{l+1} \Rightarrow \langle \eta_s : s \leqq l\rangle \in T_l$.

Therefore, by the Köning lemma there exists $\bar{\eta} \in V$ such that $\bar{\eta}$ obeys f_0 and for every $l \in \omega$, $\bar{\eta} \restriction l \in T_l$.

(c) $\bar{\eta}$ says that $\langle \bar{\eta}^\alpha : \alpha < \omega_1\rangle$ is not big (obviously this gives the contradiction).

PROOF. (c) Let q be k and $\alpha \in A$. So let $m \in \omega$ be such that $\rho \subset \eta_m^{\delta_\alpha^2, i_\alpha}$, then there exists $q_\alpha \geqq p_\alpha$ such that

$$q_\alpha \Vdash \langle \tilde{\eta}_s : s \leqq m+1 \rangle = \langle \eta_s : s \leqq m+1 \rangle.$$

Now using (iii) it should be true that

$$\rho \subset \eta_s \subseteq \eta^{\delta_\alpha^2, i_\alpha}$$

because $q_\alpha \geqq p_\alpha$. ■

1.18. LEMMA. *Let $\langle P_j ; \tilde{Q}_j : j < \gamma \rangle$ be a γ-stage iterated forcing notion with finite support and let $\langle \langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ be big. Assume that*

(i) $(\forall j < \gamma)(\Vdash_{P_j}$ "$\tilde{Q}_j \models$ *c.c.c.*") *and*

(ii) *for each* $j < \gamma$

$$\Vdash_{P_j} \text{“}\langle \langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle \text{ is big”}.$$

Then $\Vdash_{P_\gamma}$ "$\langle \langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ *is big*", *where* $P_\gamma = \varinjlim P_i$.

PROOF. Clearly the only non-trivial case is when $\mathrm{cf}(\gamma) = \omega$. (Otherwise the counterexample appears before γ.) So w.l.o.g. $\gamma = \omega$. Let $\tilde{\eta}$ be a P_γ-name for a counterexample. Therefore

(i) $\Vdash_{P_\gamma}$ "$\tilde{\eta}$ obeys f_0".

(ii) $(\exists p \in P_\gamma)(p \Vdash$ "$\tilde{\eta}$ is a counterexample").

Therefore there exists $\langle p_\alpha : \alpha \in A \rangle \subseteq P_\gamma$ satisfying

(i) $|A| = \aleph_1$ and for each $\alpha \in A$, $p \leqq p_\alpha$.

(ii) For each $\alpha \in A$, there exists $k_\alpha \in \omega$ such that $\alpha < i_\alpha$ and δ_α^1, δ_α^2 and

$$\text{“}p_\alpha \Vdash (\forall l)(k_\alpha < l < \omega \wedge \eta_{k_\alpha}^{\delta_\alpha^1, i_\alpha} \subset \eta_l^{\delta_\alpha^2, i_\alpha} \Rightarrow (\exists s \in \omega)(\eta_{k_\alpha}^{\delta_\alpha^1, i_\alpha} \subset \eta_s \subseteq \eta^{\delta_\alpha^2, i_\alpha}))\text{”}.$$

(iii) W.l.o.g. $k_\alpha = k$ and $\eta_k^{\delta_\alpha^1, i_\alpha} = \rho$.

We recall that the support of P_γ is finite, so

(iv) W.l.o.g. there is $t \in \omega$ such that $\alpha \in A$ implies $p_\alpha \in P_t$.

Again we may assume $t = 0$, because we can pass to $V[G_t]$ with G_t containig uncountably many p_α's. Now, let $p^l \in P_\gamma$ be such that

(a) $p^l < p^{l+1}$ for $l \in \omega$,

(b) $p^l \Vdash_{P_\gamma}$ "$\tilde{\eta} \upharpoonright l = \eta^l$".

Therefore $\bigcup \eta^l = \bar{\eta}$ belongs to V and obeys f_0.

(c) $\bar{\eta}$ says that $\langle \langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ is not big in V, so is a contradition. ■

The next goal is to show that using a σ-centered forcing notion, we can extend the domain of a big family in such a way that the new sequences obey some given function f. Guaranteed this, we will iterate ω_2-times, and we will get that the underlying functions for the big family are a dominating family.

1.19. LEMMA. *Let V be a model of ZFC, and let $f \in \omega^\omega$ and let*

$$\langle\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$$

be in V satisfying

(i) $\beta < \omega_2$,

(ii) $\delta < \gamma < \beta \rightarrow \bar{\eta}^{\delta,i} <^* \bar{\eta}^{\gamma,i}$, *for* $i < \omega_1$,

(iii) $\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle$ *obeys* f_δ,

(iv) $\delta < \gamma < \beta \rightarrow f_\delta \leqq^* f_\gamma \leqq^* f$, *and*

(v) $\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle$ *is big.*

Then there exists a σ-centered forcing notion Q adding a sequence $\langle \bar{\eta}^{\beta,i} : i < \omega_1 \rangle$ satisfying

(a) $\Vdash_Q$ "$\langle\langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta + 1 \rangle$ *is big*",

(b) $\Vdash_Q$ "$(\forall i < \omega_1)(\bar{\eta}^{\gamma,i} <^* \bar{\eta}^{\beta,i})$", *for each* $\gamma < \beta$.

PROOF. Let Q^i be the following forcing notion $\langle a, \bar{\nu}, h \rangle \in Q^i$ if

(α) there exists $k < \omega$ such that $a \subseteq k$,

(β) $\bar{\nu} = \langle \nu_l : l \in a \rangle$ and for every $l \in a$, $\nu_l \in 2^{f(l)}$,

(γ) h is a function satisfying

$$|\mathrm{dom}(h)| < \aleph_0 \quad \text{and} \quad h : \omega \times \beta \rightarrow \omega$$

and for every $(l, \gamma) \in \mathrm{dom}(h)$, if $n \in a$ and $h(l, \gamma) < n$ then there exists $m \in \omega - l$ such that $\eta_m^{\gamma,i} \subseteq \nu_n$.

We say that $\langle a^1, \bar{\nu}^1, h^1 \rangle \leqq_{Q^i} \langle a^2, \bar{\nu}^2, h^2 \rangle$ if $a^1 \subseteq a_2^2$ and $\bar{\nu}^1 \subseteq \bar{\nu}^2$ and $h^1 \subseteq h^2$. This completes the definition of Q^i.

FACT. Q^i is σ-centered.

PROOF. We may consider ω having the discrete topology. Then ω is clearly separable. Also $|\beta| \leqq \aleph_0$ therefore ${}^{\omega\times\beta}\omega$ is also a separable space. Let $\langle F_j : j < \omega \rangle$ be dense in ${}^{\omega\times\beta}\omega$. For each $j < \omega$ we define the following relation on Q^i: $\langle a^1, \bar{\nu}^1, h^1 \rangle \sim_j \langle a^2, \bar{\nu}^2, h^2 \rangle$ if $a^1 = a^2$ and $\bar{\nu}^1 = \bar{\nu}^2$ and F_j belong to $[h_1] \cap [h_2]$, where for $h : \omega \times \beta \rightarrow \omega$, $|\mathrm{dom}(h)| < \aleph_0$, we set $[h] = \{F \in {}^{\omega\times\beta}\omega : F \restriction \mathrm{dom}(h) = h\}$. Clearly this relation will give a ω-partition of Q^i into directed sets. ■

FACT. Let $H \subseteq Q^i$ be generic over $V[G]$, and let

$$\eta^{\beta,i} = \bigcup \{\bar{\mu} : (\exists \langle a, \bar{\nu}, h \rangle \in H)(\bar{\mu} = \bar{\nu})\}.$$

Then for every $\gamma < \beta$, $\eta^{\gamma,i} <^* \eta^{\beta,i}$.

PROOF. Fix $\langle a, \bar{v}, h \rangle$ in Q^i, and $\gamma < \beta$. Let $l \in \omega$ and let $m > \sup(a)$. Then we can extend $\langle a, \bar{v}, h \rangle$ to $\langle a, \bar{v}, h \cup \{(\langle l, \gamma \rangle, m)\} \rangle$ and this condition forces that for every $p > m$, $p \in \mathrm{dom}(\eta^{\beta,i})$ there exists $q \in \omega - l$ such that

$$\eta_q^{\gamma,i} \subseteq \eta_p^{\beta,i}. \qquad \blacksquare$$

FACT. Let $\bar{\eta}$ be in V, $\bar{\eta}$ obeys f_0. Let $i < \omega_1$ satisfying

For every $\delta_1 \leqq \delta_2 < \beta$ there is no $q \in \omega$ satisfying

$$(\forall p)(q < p < \omega \wedge \eta_q^{\delta_1,i} \subset \eta_q^{\delta_2,i} \rightarrow (\exists s \in \omega)(\eta_q^{\delta_1,i} \subset \eta_s \subseteq \eta_q^{\delta_2,i})).$$

Then in V^{Q_i} the following holds: for every $\delta_1 \leqq \delta_2 \leqq \beta$ there is no $q \in \omega$ satisfying

$$(\forall p)(q < p < \omega \wedge \eta_q^{\delta_1,i} \subset \eta_q^{\delta_2,i} \rightarrow (\exists s \in \omega)(\eta_q^{\delta_1,i} \subset \eta_s \leqq \eta_q^{\delta_2,i})).$$

PROOF. *Case* 1. $\delta_1 \leqq \delta_2 < \beta$. By assumption and 1.16.

Case 2. $\delta_1 < \delta_2 = \beta$. Let $q \in \omega$ be given and let $\langle a, \bar{v}, h \rangle$ be in Q^i. Let $\gamma_1 < \gamma_2 < \cdots < \gamma_n$ be a list of all the ordinals which appear in the domain of h. Then w.l.o.g. $\delta_1 \leqq \gamma_n$. Using 1.10(b) we pick $q < l_1 < l_2 < \cdots < l_n$ such that

(*) for each j, if $p \in \omega - l_{j+1}$ then there is $r \in \omega - l_j$ satisfying

$$\eta_r^{\gamma_j,i} \subset \eta_p^{\gamma_{j+1},i}.$$

Now applying n-time our assumption on i, using δ_1 and γ_n, we get ρ satisfying

(α) $\eta_q^{\delta_1,i} \subset \rho$,

(β) for no $s \in \omega$, $\eta_q^{\delta_1,i} \subset \eta_s \subset \rho$,

(γ) $|\rho| > f_{\gamma_n}(l_n)$.

Now by a bookkeeping argument we can extend ρ to v_m and v_m satisfying (β). Now using (*) we see that $\langle a \cup \{m\}, \bar{v} \cup \{v_m\}, h \rangle$ is a condition, forcing that q satisfies the condition for δ_1 and δ_2. ■

REMARK. We can use the same argument to see that "a" is infinite in the generic extension.

Now we define Q to be

$$Q = \prod_{i < \omega_1} Q^i$$

with finite support, and we order Q by taking the order induced by the product.

FACT. Q is σ-centered.

PROOF. Like the proof of $Q_i \models \sigma$-centered. ■

FACT. Let $\langle \bar{\eta}^{\beta,i} : i < \omega_1 \rangle$ be the Q-name of the sequence given by forcing with Q. Then $\Vdash_Q$ "$\langle \langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta + 1 \rangle$ is big".

PROOF. Clearly $\Vdash_Q$ "$\bar{\eta}^{\beta,i}$ obeys f". Let $\bar{\nu}$ be a Q-name of a sequence such that $\Vdash_Q$ "$\bar{\nu}$ obey f_0". Then, by c.c.c., there exists $\xi < \omega_1$ such that

$$\bar{\nu} \text{ is a } \left(\prod_{i<\xi} Q^i \right)\text{-name.}$$

Therefore we have that in $V^{\prod_{i<\xi} Q^i}$ $\langle \langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta < \beta \rangle$ is big, so we can use the above fact and that the forcing Q^i is absolute. ■

Now we are ready to give the forcing definition.

1.20. DEFINITION. P_{ω_2} is good if there is $\bar{Q} = \langle P_\alpha : \underset{\sim}{Q}_\alpha : \alpha < \omega_2 \rangle$, an ω_2-stage iterated forcing notion satisfying

(0) P_{ω_2} is the direct limit of $\bar{Q}$.

(1) If $\alpha = 0$ or $\mathrm{cof}(\alpha) = \omega_1$ then

$$\Vdash_{P_\alpha} \text{“}\underset{\sim}{Q}_\alpha \text{ is Hechler real forcing”.}$$

(Hechler real forcing is $\{(n, f) : n \in \omega \wedge f \in \omega^\omega\}$ with the obvious order to get a dominating function.) To make the construction clear, let $\langle \alpha_\beta : \beta < \omega_2 \rangle$ be $\{0\} \cup \{\alpha : \mathrm{cof}(\alpha) = \omega_1 \text{ and } \alpha < \omega_2\}$. Let $\underset{\sim}{f}_\beta$ be the $P_{\alpha_\beta+1}$-name of the Hechler real added by $P_{\alpha_\beta+1}/P_{\alpha_\beta}$ over $V^{P_{\alpha_\beta}}$.

(2) Suppose we have $\langle P_\alpha ; \underset{\sim}{Q}_\alpha : \alpha < \alpha_\beta \rangle$ for some $\beta < \omega_2$ then for every $\delta < \gamma \leqq \beta$ the following hold:

(i) $\Vdash_{P_{\alpha_\gamma+1}}$ "$\underset{\sim}{Q}_{\alpha_\gamma+1}$ adds $\langle \bar{\eta}^{\gamma,i} : i < \omega_1 \rangle$ obeying $\underset{\sim}{f}_\gamma$,

(ii) $\Vdash_{P_{\alpha_\gamma+1}}$ "$\underset{\sim}{Q}_{\alpha_\gamma+1}$ is σ-centered",

(iii) $\Vdash_{P_{\alpha_\gamma+2}}$ "$(\forall i < \omega_1)(\bar{\eta}^{\delta,i} <^* \bar{\eta}^{\gamma,i})$.

(3) For every $\beta < \omega_2$ we define $\langle \underset{\sim}{x}_\beta^i : i < \omega_1 \rangle$ such that for every $i < \omega_1$

(i) $\underset{\sim}{x}_\beta^i$ is a $P_{\alpha_\beta+2}$-name of a member of 2^ω,

(ii) $\Vdash_{P_{\alpha_\beta+2}}$ "$\underset{\sim}{x}_\beta^i \in A(\bar{\eta}^{\beta,i})$".

(*) Let $\langle \underset{\sim}{\nu}_\beta : \beta < \omega_2 \rangle$ be such that for every $\beta < \omega_2$, $\bar{\nu}_\beta$ is a $P_{\alpha_\beta+1}$-name satisfying

$$\Vdash_{P_{\alpha_\beta+1}} \text{“}\bar{\nu}_\beta \text{ obeys } \underset{\sim}{f}_0\text{”.}$$

And for every P_{ω_2}-name $\bar{\nu}$ such that

$$\Vdash_{P_{\omega_2}} \text{“}\bar{\nu} \text{ obeys } \underset{\sim}{f}_0\text{”}$$

there exists $\beta < \omega_2$ satisfying

$$\Vdash_{P_{\omega_2}} \text{“}\bar{\underset{\sim}{\nu}} = \bar{\underset{\sim}{\nu}}_\beta\text{”}.$$

Then

(iii) there exists $i < \omega_1$ such that

$$\Vdash_{P_{\alpha_\beta+2}} \text{“}\underset{\sim}{x}^i_\beta \notin A(\bar{\underset{\sim}{\nu}}_\beta)\text{”}.$$

(4) For each $\gamma < \omega_1$

$$\Vdash_{P_\gamma} \text{“}\langle \langle \bar{\underset{\sim}{\eta}}^{\delta,i} : i < \omega_1 \rangle : \alpha_\delta < \gamma \rangle \text{ is big”}.$$

(5) $\Vdash_{P_{\omega_2}}$ “$MA(\sigma$-centered)”.

1.21. Lemma. *If $V \models GCH$, then there is P_{ω_2} which is good.*

Proof. Let

$$A = \{0\} \cup \{\beta < \omega_2 : \mathrm{cof}(\beta) = \omega_1\} = \{\alpha_\beta : \beta < \omega_2\},$$

$$B = \{\alpha + 1 : \alpha \in A\}.$$

Clearly $\omega_2 - A \cup B$ has cardinality $\aleph_2$, so by a bookkeeping argument and using ordinals from $\omega_2 - A \cup B$ we can easily get (5). We get (0) and (1) easily. For (2) we use Lemma 1.19. For (3) we give a partition of ω_2 into $\langle B_i : i < \omega_2 \rangle$, satisfying $B_i \in [\omega_2]^{\omega_2}$ for each $i < \omega_2$. Then in order to get $\langle \underset{\sim}{\nu}_\beta : \beta < \omega_2 \rangle$ satisfying (*), we use B_β ($\beta < \omega_2$), to take care of each possible $\bar{\underset{\sim}{\nu}}$ obeying f_0 which is in V^{P_β} (w.l.o.g. $B_\beta \cap \beta + 1 = \emptyset$). (This means that if $\gamma \in B_\beta$, then in the stage α_γ we take care of the member γ of list of $\bar{\underset{\sim}{\nu}}$'s.)

Now we know that $\bar{\underset{\sim}{\nu}}_\beta \in V^{P_{\alpha_\beta}}$ and therefore since $\langle \langle \bar{\eta}^{\delta,i} : i < \omega_1 \rangle : \delta \leqq \beta \rangle$ is big in $V^{P_{\alpha_\beta+1}}$, we can pick $\langle \underset{\sim}{x}^i_\beta : i < \omega_1 \rangle$ satisfying the required conditions. Remember that in $V^{P_{\alpha_\beta+1}}$ the following holds:

(i) $(\forall i < \omega_1)(\forall \delta < \beta)(\bar{\underset{\sim}{\eta}}^{\delta,i} <^* \bar{\underset{\sim}{\eta}}^{\beta,i})$,

(ii) $A(\underset{\sim}{\eta}^{\beta,j}) - A(\bar{\underset{\sim}{\nu}}_\beta) \neq \emptyset$, for almost every j.

Last we need to check (4). We have three cases, and we use induction.

Case 1. $\{\alpha_\delta : \alpha_\delta < \gamma\}$ is bounded in γ. In this case (4) follows by using 1.16 and 1.18.

Case 2. $\{\alpha_\delta : \alpha_\delta < \gamma\}$ is unbounded in γ and $\mathrm{cf}(\gamma) = \omega_1$.

This case is trivial, because every possible counterexample appears in an intermediate stage, so the inductive hypothesis applies.

Case 3. $\{\alpha_\delta : \alpha_\delta < \gamma\}$ is unbounded in γ and $\mathrm{cf}(\gamma) = \omega$.

The proof of this is like 1.18 and we leave it to the reader. The only new assumption is that for each $\gamma \in A$, $\{\delta^1_\gamma, \delta^2_\gamma\} \subseteq \sup(p_\gamma)$. Then we use the induction hypothesis in V^{P_i}. ■

1.22. THEOREM. Cons(ZF)$\Rightarrow$Cons($ZFC + MA(\sigma$-centered$) + \neg$Add(SMZ)).

PROOF. Let $V = L$, and let P_{ω_2} be good. We use the notation of 1.20 and 1.21.

Let $\underset{\sim}{X}_i$ be a P_{ω_2}-name satisfying

$$\Vdash_{P_{\omega_2}} \text{“}\underset{\sim}{X}_i = \{\underset{\sim}{x}^i_\beta : \beta < \omega_2\}\text{”}.$$

Then we have

(i) $\Vdash_{P_{\omega_2}}$ “$\underset{\sim}{X}_i \in SMZ$” by 1.20(3)(ii), 1.20(2)(iii) and 1.13(b).

(ii) $\Vdash_{P_{\omega_2}}$ “$\bigcup_{i<\omega_1} \underset{\sim}{X}_i \notin SMZ$” by 1.20(3)(iii).

(iii) $\Vdash_{P_\omega}$ “$MA(\sigma$-centered)” by 1.20(5). ■

1.23. REMARK. Add(SMZ) does not imply Add($\mathscr{B}$).

PROOF. By Pawlikowski [9], there exists a model for Add(SMZ) where $b = \aleph_1$. ■

§2. *MA*(σ-centered) and the Dual Borel Conjecture

We recall

2.0. DEFINITION. (a) A set of reals X is strongly meager if for every measure zero set M there exists a real number x such that $(X + x) \cap M = \varnothing$.

(b) Dual Borel Conjecture: Every strongly meager set is countable.

It is a very interesting open question if the collection of the strongly meager set is an Ideal.

2.1. THEOREM (T. Carlson). Cons(ZFC) *implies* Cons(ZFC + *Dual Borel Conjecture*).

We proved in Ihoda–Shelah [4] the following:

2.2. THEOREM. *The following theories are equiconsistent*:

(1) *ZFC + there exists an inaccessible cardinal.*

(2) *ZFC + Borel Conjecture + every Σ^1_2-set of reals has the property of Baire.*

(3) *ZFC + Borel Conjecture + every Σ^1_2-set of reals is Lebesgue measurable.*

(4) *ZFC + Dual Borel Conjecture + every Σ^1_2-set of reals is Lebesgue measurable.*

And we asked if

(5) *ZFC + Dual Borel Conjecture + every Σ^1_2-set of reals has the property of Baire has the same force as* (1).

The following theorem gives a negative answer to this question.

2.3. THEOREM. *The following theories are equiconsistent*:
(i) *ZFC*.
(ii) *ZFC + MA(σ-centered) + Dual Borel Conjecture*.

PROOF. Essentially we all give a generalization of Carlson's proof of 2.1. (ii)$\Rightarrow$(i) is clear.

The rest of this section is devoted to the proof of (i)$\Rightarrow$(ii).

2.4. DEFINITION. For a real c, Pc is the poset of sets $A \subseteq [0, 1]$ of measure $> c$ which are finite unions of closed rational intervals. Order Pc by inverse inclusion.

2.5. FACT. Forcing with Pc is equivalent to forcing to add a Cohen real.

2.6. THEOREM. *There exists a constant c such that if $A \subseteq \mathbf{Z}/n\mathbf{Z}$ for some $n \geqq 2$ and $|A| > 2$ there exists $B \subseteq \mathbf{Z}/n\mathbf{Z}$ with $|B| < cn \log|A|/|A|$ such that $A + B = \mathbf{Z}/n\mathbf{Z}$.*

PROOF. Lorentz [6].

2.7. COROLLARY. *If $X \subseteq [0, 1]$ has cardinality $k \geqq 2$ then there exists an open set $U \leqq [0, 1]$ such that $[0, 1] = X + U \ (\mathrm{mod}\ \mathbf{Z})$ and $\mu(U) < 4(\log k)/k$.*

PROOF (Carlson [2]). Note that U is a finite union of rational open intervals. ■

2.8. LEMMA. *Let $Pc*\underset{\sim}{Q}$ be a σ-centered forcing notion. Let $X \subseteq [0, 1]$ be uncountable. If $G \subseteq Pc*\underset{\sim}{Q}$ is generic over V and K is the compact set of measure c constructed from $G \cap Pc$, then in $V[G]$ no translate of K has uncountable intersection with X.*

PROOF. Suppose that there exists a $Pc*\underset{\sim}{Q}$-name of a real number $\underset{\sim}{r}$, such that

$$0 \Vdash_{Pc*\underset{\sim}{Q}} \text{“}(\underset{\sim}{r} + X) < \underset{\sim}{K} \text{ is uncountable”}.$$

Therefore there exists $x_i \in X$, $\langle p_i, \underset{\sim}{q}_i \rangle \in Pc*\underset{\sim}{Q}$ for $i < \omega_1$ such that

$$\langle p_i, \underset{\sim}{q}_i \rangle \Vdash \text{“}x_i + \underset{\sim}{r} \in \underset{\sim}{K}\text{”}.$$

Pc is countable, and $Pc*\underset{\sim}{Q}$ is σ-centered, therefore without loss of generality we can assume that for every i, $p_i = p$, fixed, and for $i_1, \ldots, i_n < \omega_1$, $\langle p, \underset{\sim}{q}_{i_1} \cup \cdots \cup \underset{\sim}{q}_{i_n} \rangle$ is an upper bound for $\langle p, \underset{\sim}{q}_{i_j} \rangle$, $1 \leqq j \leqq n$.

2.9. Claim. There exists $p' \geqq p$ and $i_1, \ldots, i_n$ such that

$$\{x_{i_1}, \ldots, x_{i_n}\} + \sim p' = [0, 1] \quad (\text{mod } \mathbf{Z}).$$

Proof. Using 1.7 we pick $i_1, \ldots, i_n$, and U such that
(i) $\{x_{i_1}, \ldots, x_{i_n}\} + U = [0, 1]$ (Mod Z),
(ii) $\mu(\sim U \cap p) > c$. ■

Now if $p', i_1, \ldots, i_n$ are as in 2.9, we have that

$$\langle p', \underset{\sim}{q}_{i_1} \cup \cdots \cup \underset{\sim}{q}_{i_n} \rangle \Vdash \text{“}\underset{\sim}{r} + \{x_1, \ldots, x_n\} \subseteq \underset{\sim}{K}$$

and

$$\{x_1, \ldots, x_n\} + \sim \underset{\sim}{K} = [0, 1]\text{”},$$

and this is a contradiction. This finishes the proof of 2.8. ■

2.10. Remark. From 2.8, using the fact that adding one Cohen real is isomorphic to adding ω-Cohen reals, we obtain that there exists K_n, a compact set of measure $1 - 1/n$ such that if $C = \bigcup_n K_n$ then C has measure one and in $V[G]$ no translate of C has uncountable intersection with X.

Now we conclude the proof of 2.3. We will give the usual forcing notion that forces MA(σ-centered) over a model of GCH. Therefore we will assume $V \vDash GCH$.

It is well known that there exists $\bar{Q} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \omega_2 \rangle$, a ω_2-stage iterated forcing satisfying

(1) for every $\alpha < \omega_2$ there exists $\beta > \alpha$,

$$P_\beta \Vdash \text{“}\underset{\sim}{Q}_\beta \text{ is Cohen real forcing”};$$

(2) for every $\alpha < \beta < \omega_2$,

$$P_\alpha \Vdash \text{“}\underset{\sim}{P}_{\alpha-\beta} \text{ is } \sigma\text{-centered”};$$

(3) $P_{\omega_2} \Vdash$ “MA(σ-centered)”.
Therefore, using the countable chain condition of P_{ω_2}, 2.10, and (1), (2) we can conclude

(4) $P_{\omega_2} \Vdash$ “Dual Borel Conjecture”.

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